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SELECTIVITY FUNCTIONS AND A TWO-CONTOUR ALGORITHM AS A TOOL USED IN THE PROBLEMS OF SYNTHESIS OF MULTILAYER CONSTRUCTIONS

A. Yu. Bushuev and V. V. Gorskii

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An efficient iteration technique is suggested to solve the problem of selecting the thicknesses of a multilayer construction subjected to radiative-convective fluxes from the condition providing prescribed temperatures at controlled points.

In [1, 2], a new approach was formulated for the implementation of iteration processes of solving radiative gasdynamics problems characterized by the need for carrying out a great number of complicated calculations of the same type. Essentially, this approach consists in implementing a double-contour iteration scheme, in the inner contour of which a great number of calculations of the same type are carried out for a radically simplified analog of the initial formulation of the problem. In the outer contour, the approximate algorithm is corrected based on the solution of the problem in a rigorous formulation.

As the long-standing experience of using the double-contour algorithm for solving a wide spectrum of physicomathematical problems at the Scientific Production Association of Mechanical Engineering has shown, its use is always accompanied by a qualitative decrease in the laboriousness of solving the applied problems without any loss of accuracy.

The present work illustrates the possibility of applying this approach to solving one kind of inverse heat conduction problems, namely, to determining the layer thicknesses of a one-dimensional multilayer construction (a construction packet) that correspond to the prescribed values of maximum permissible temperatures at some controlled points with the known thermophysical properties of materials and laws of thermal loading of the packet.

The first results of solving the indicated problem by this technique are published in [3]. However, the method used there for correction of the approximate formulation of the problem in the inner contour is rather artificial although its use provides a stable solution of the problem set up.

We suggest another, more natural approach to solution of the indicated problem.

Consider the following design problem. Let the layer thicknesses of a construction packet subjected to the action of a high-temperature medium be determined from the condition providing the equality of the temperatures at the boundaries of the layers to the prescribed values:

$$\varphi_i^* (h_1, h_2, \dots , h_m) = T_i^* (h_1, h_2, \dots , h_m) - \hat{T}_{I(i)} = 0 , \quad i = \overline{1, m} , \quad I (i) \in [0, n] , \tag{1}$$

where

$$T_i^* = T (x_i, \tau^*) = \max_{\tau \in [0, \hat{\tau}]} T (x_i, \tau) .$$

In this case, the temperature regime of the construction is found from the solution of the following boundary-value problem:

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$$\rho_k c_k (T) \frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x} \left(\lambda_k (T) \frac{\partial T}{\partial x} \right), \quad x_{k-1} < x < x_k, \quad 0 < \tau \le \hat{\tau}, \quad k = \overline{1, n}, \tag{2}$$

$$T(x, 0) = T_0, \quad x_0 \le x \le x_n, \tag{3}$$

$$T(x_k - 0, \tau) = T(x_k + 0, \tau), \quad k = \overline{1, n-1}, \quad \tau > 0, \tag{4}$$

$$\lambda_k (T) \frac{\partial T(x_k - 0, \tau)}{\partial x} = \lambda_{k+1} (T) \frac{\partial T(x_k + 0, \tau)}{\partial x}, \quad k = \overline{1, n-1}, \quad \tau > 0. \tag{5}$$

The boundary conditions on the surfaces w_0 and w_n of the construction packet (CP) in the general case have the form

$$-\lambda_1 (T) \frac{\partial T(x_0, \tau)}{\partial x} = q_{w_0} (T), \tag{6}$$

$$\lambda_n (T) \frac{\partial T(x_n, \tau)}{\partial x} = q_{w_n} (T). \tag{7}$$

A solution of the formulated problem is sought within the framework of the double-contour iteration algorithm. The problem of selecting the thicknesses is solved in the inner contour where use is made of a rought mathematical model that differs from (2)-(7) by employing fixed thermophysical characteristics for each layer of the packet. Furthermore, a solution of the heating problem is sought here for a substantially smaller number of nodal values of the sought functions. This mathematical model is corrected in the outer contour on the basis of the solution of the initial formulation of problem (2)-(7) for the layer thicknesses found in the inner contour. It must provide the identity of temperatures at the nodes of the rough mathematical model found for the same thicknesses of the packet layers in the initial and simplified formulations.

The indicated problem can be solved by different methods. Thus, in [3], it has been solved by employing, in the inner contour, maximum permissible temperatures differing from T_i^*, which are calculated by a rather complicated algorithm. In the present work, this problem is solved in a more natural way: by introducing the additional correction function $Q(\bar{x}, \tau)$ into an analog of Eq. (2) for the inner contour, by which we understand the discrepancy of the heat conduction equation solved in the inner contour for the function $T^{(2)}(\bar{x}, \tau)$ solved in the outer contour, i.e., Eq. (2) reduces to the form

$$\rho_k \bar{c}_k \frac{\partial T}{\partial \tau} = \frac{\bar{\lambda}_k}{h_k^2} \frac{\partial^2 T}{\partial \bar{x}^2} + Q(\bar{x}, \tau), \tag{8}$$

where

$$\bar{x} = \frac{x - x_{k-1}}{x_k}, \quad k = \overline{1, n}; \quad Q(\bar{x}, \tau) = \rho_k \bar{c}_k \frac{\partial T^{(2)}}{\partial \tau} - \frac{\bar{\lambda}_k}{h_k^2} \frac{\partial^2 T^{(2)}}{\partial \bar{x}^2}. \tag{9}$$

In the outer contour, use is made of the mathematical model (2)-(7) that allows for the temperature dependence of thermophysical properties and of a fine-mesh grid in solving the heating problem.

The layer thicknesses satisfying the nonlinear equation (1) are sought by a modified Newton method. In this case, the tool of sensitivity functions is used to calculate partial derivatives of the temperatures with respect to the thicknesses of varied layers at controlled nodes [4].

Differentiating Eq. (8) with respect to the thickness of a varied layer, we arrive at a system of linear differential equations for the sensitivity functions $\varphi_{h,j}$, by which we understand the derivatives of the packet temperatures with respect to the thicknesses of the varied layers:

$$\bar{\rho}_k \bar{c}_k \frac{\partial \varphi_{h,j}}{\partial \tau} = \bar{\lambda}_k \frac{\partial^2 T}{\partial \bar{x}^2} - \frac{2\delta_{j,k}}{h_k^3} \bar{\rho}_k \bar{c}_k \frac{\partial T}{\partial \tau}, \quad j = \overline{1, m}; \quad k = \overline{1, n}, \tag{10}$$

where

$$\delta_{j,k} = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

Differentiating relations (6) and (7) with respect to the thickness of the varied layer h_j at fixed thermal conductivities λ, we obtain the following boundary conditions for the sensitivity functions:

$$-\bar{\lambda}_1 \frac{\partial \varphi_{h,j}}{\partial \bar{x}} = q_{\mathrm{w}_0, T}(T)\, \varphi_{h,j} - \frac{\bar{\lambda}_1 \delta_{j,1}}{h_1^2} \frac{\partial T}{\partial \bar{x}}, \quad j = \overline{1, m}; \tag{11}$$

$$\bar{\lambda}_n \frac{\partial \varphi_{h,j}}{\partial \bar{x}} = q_{\mathrm{w}_n, T}(T)\, \varphi_{h,j} - \frac{\bar{\lambda}_n \delta_{j,n}}{h_n^2} \frac{\partial T}{\partial \bar{x}}, \quad j = \overline{1, m}. \tag{12}$$

A search for a solution of the problem under consideration consists in executing the following sequence of operations at each outer pth iteration:

Step 0. Choose the initial approximation $h^0 = \{h_1^0, h_2^0, ..., h_m^0\}$ and the parameters ε_1 and ε_2. Assume that $Q(\bar{x}, \tau) = 0$.

Step 1. Determine the thicknesses of the varied layers $h_j^{(l)}$ from the system of linear algebraic equations

$$\sum_{j=1}^{m} \varphi_{h,i,j}^{*(l)} \Delta h_j^{(l)} = -\varphi_i^{*(l)}, \quad i = \overline{1, m}, \tag{13}$$

obtained by linearization of the system of equations (1).

For passing from the lth to the $(l+1)$th iteration, use is made of formulas of the type

$$h_j^{(l+1)} = h_j^{(l)} + \beta_l \Delta h_j^{(l)}, \quad j = \overline{1, m}, \quad l = 1, 2, \dots, \tag{14}$$

where

$$\beta_l = \frac{1}{\max\left\{1, \frac{1}{\Delta} \max_j \left(\frac{|\Delta h_j^{(l)}|}{h_j^{(l)}} \right)\right\}}.$$

The iteration process in the inner contour is completed provided

$$|\varphi_i^{*(l)}| \le \varepsilon_1, \quad i = \overline{1, m}. \tag{15}$$

The functionals $\varphi_i^{*(l)}$ are calculated using the approximate mathematical model.

Step 2. Calculation of the temperature field $T^{(p)}(x, \tau)$ using the mathematical model (2)-(7) and determination of temperatures at the nodes of the rough grid.

Step 3. Checking of the condition of iteration completion

TABLE 1. Properties of Materials

No. of material k	Dependence		Averaged values		Density, ρ_k
	$\lambda_k(T)$	$c_k(T)$	$\bar{\lambda}_k$	$\bar{c}_k$	
1	$30-5\bar{T}$	$1500+0.2T$	21	1900	2100
2	$(1+\bar{T}+\bar{T}^2)\cdot 0.005$	$800+0.2T$	0.1	1000	200
3	$0.04+0.01\bar{T}+0.03\bar{T}^2$	$1800-T+400\bar{T}^2$	0.06	1300	100
4	$0.04+0.1\bar{T}$	$-400+5T$	0.45	1400	1800

TABLE 2. Convergence of the Process

Iteration No.	$100h_1$	$100h_2$	$100h_3$	φ_1^*	φ_2^*	φ_3^*	$\Sigma\,\lvert\varphi_i^*\rvert$
1	17.6	12.3	40.2	39	245	8	292
2	20.5	18.4	39.6	16	50	−6	72
3	21.8	19.7	35.7	6	5	1	12
4	22.3	19.8	35.1	2	0.45	0.4	2.85

$$|T_i^{*(p)} - \hat{T}_{l(i)}| \le \varepsilon_2 , \quad i = \overline{1, m} . \tag{16}$$

If the condition is fulfilled, the thicknesses of the varied layers determined at step 1 can be assumed to be the final solution of the problem of synthesis, otherwise we pass to the following step.

Step 4. Determination of the correction function (9) and passage to step 1.

Construction of numerical solutions both of the primal heat conduction problem and the problem on determination of the sensitivity functions is accomplished in the present work by using the implicit difference schemes for Eqs. (2) and (8) and the factorization method [5].

According to the algorithm described, we performed numerical calculations of a number of methodical problems differing in the choice of the initial approximation and the structure of construction packets and in the mode of thermal loading.

As an illustration of the investigations performed, the solution for the model problem from [4] is given.

It is required to select the thicknesses of three layers nearest to the heated surface w_0 of the CP so that the temperatures at the boundaries of the layers are equal to the prescribed values:

$$\hat{T}_1 = 1773 \text{ K}, \quad \hat{T}_2 = 1273 \text{ K}, \quad \hat{T}_3 = 343 \text{ K}.$$

The thermophysical properties of layer materials are given in Table 1.

The initial thicknesses of the varied layers are assumed to be: h_1 = 5 mm, h_2 = 5 mm, and h_3 = 10 mm. The thickness of the fourth layer is fixed and equal to 3 mm.

Heat fluxes on the boundary surfaces w_0 and w_n of the CP are calculated by the following formulas

$$q_{w_0} = 0.03 \exp\left[\left(\frac{3(\tau-1000)}{1000} - 2\right)\left(\frac{\tau-500}{500}\right)^2\right](H_{r,w_0} - H_{w_0}) - \varepsilon_{w_0}\sigma T_{w_0}^4 ,$$

where

$$H_{w_0} = 954 T_{w_0} + 0.0862 T_{w_0}^2 ,$$

$$q_{w_n} = 15\,(323 - T_{w_n}) + \varepsilon_{w_n}\sigma\,(333^4 - T_{w_n}^4) .$$

The convergence of the iteration process in the outer contour is illustrated by the calculated results given in Table 2.

An analysis of the above calculations shows the effectiveness of the double-contour algorithm, with the aid of which the solution of the inverse design problem is constructed with the required accuracy and with limited use of a rigorous mathematical model.

NOTATION

T, temperature, K; φ_i^*, functional-discrepancy between the maximum temperature at the ith controlled node T_i^* and its maximum permissible value $\widehat{T}_{I(i)}$, K; i, number of limitation; $I(i)$, number of the boundary of the construction packet at which the ith limitation holds; m, number of controlled temperatures (the number of varied thicknesses); n, number of layers in the CP; $\varphi_{h,j} = \partial T/\partial h_j$, sensitivity function; $\varphi_{h,i,j} = \varphi_{h,j}(x_i, \tau)$, sensitivity function at the x_i-nodes; $T^{(2)}$, temperatures calculated in the outer contour, K; T_0, initial temperature of the construction, K; $\bar{T} = T/1000$; $Q(\bar{x}, \tau)$, correction function, W/m^2; q_{w_0}, q_{w_n}, densities of the heat fluxes supplied to the boundaries w_0 and w_n of the CP, W/m^2; c, specific heat, J/(kg·K); $\bar{c}$, averaged specific heat, J/(kg·K); λ, thermal conductivity, W/(m·K); $\bar{\lambda}$, averaged thermal conductivity, W/(m·K); ρ, material density, kg/m^3; β_l, weight coefficient of the iteration process ($0 < \beta_l \leq 1$); Δ, maximum permissible relative change in the sought quantities ($\Delta = 0.2$); $h = (h_1, h_2, \ldots h_m)$, combination of varied layers, m; ε_{w_0}, ε_{w_n}, emissivity factors of the CP boundary surfaces; H_{r,w_0}, enthalpy of recovery of the gas flux on the boundary surface w_0 of the CP, J/kg; H_{w_0}, gas enthalpy at the wall temperature, J/kg; τ, time, sec; $\widehat{\tau}$, right-hand value of the time interval; τ^*, time of attaining the maximum temperature at the controlled node; ε_1, ε_2, requires accuracies of iteration processes in the inner and outer contours; l, p, current numbers of iteration in the inner and outer contours, respectively. The index on the thermophysical quantities indicates the number of the material used (see Table 1) and coincides with the number of the CP layer.

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